Model risk and Bayesian decisions for financial processes

Wei Lao

Institute for Stochastics

University of Karlsruhe

November 25, 2007

Outline

Asset price dynamics in "path space model":

- observation time $T = n$
- time set $I = \{0, 1, 2, \dots, n\}$ (in units)
- capital growth factor $Y_i = u_i$ or d_i, with $0 < d_i < e^r < u_i$
- asset price at time $t \in I$:

$$X_0 := x, \qquad X_t = x \cdot \prod_{i=1}^{t} Y_i$$

- random paths $\omega = (X_t : t \in I)$
- canonical space of all possible sample paths Ω

Asset price dynamics in "path space model":

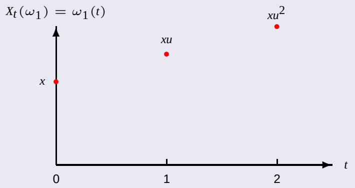
- observation time $T = n$
- time set $I = \{0, 1, 2, \ldots, n\}$ (in units)
- capital growth factor $Y_i = u_i$ or d_i, with $0 < d_i < e^r < u_i$
- asset price at time $t \in I$:

$$X_0 := x, \qquad X_t = x \cdot \prod_{i=1}^{t} Y_i$$

- random paths $\omega = (X_t : t \in I)$
- canonical space of all possible sample paths Ω

2-Period-Binomial-Model:

1. path ω_1:



Asset price dynamics in "path space model":

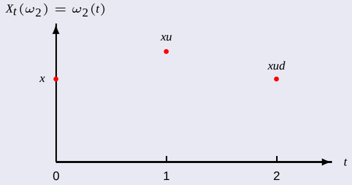
- observation time $T = n$
- time set $I = \{0, 1, 2, \ldots, n\}$ (in units)
- capital growth factor $Y_i = u_i$ or d_i, with $0 < d_i < e^r < u_i$
- asset price at time $t \in I$:

$$X_0 := x, \qquad X_t = x \cdot \prod_{i=1}^{t} Y_i$$

- random paths $\omega = (X_t : t \in I)$
- canonical space of all possible sample paths Ω

2-Period-Binomial-Model:

2. path ω_2:



Asset price dynamics in "path space model":

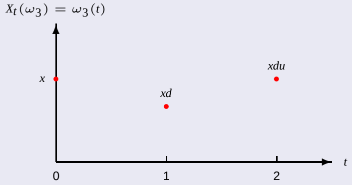
- observation time $T = n$
- time set $I = \{0, 1, 2, \ldots, n\}$ (in units)
- capital growth factor $Y_i = u_i$ or d_i, with $0 < d_i < e^r < u_i$
- asset price at time $t \in I$:

$$X_0 := x, \qquad X_t = x \cdot \prod_{i=1}^{t} Y_i$$

- random paths $\omega = (X_t : t \in I)$
- canonical space of all possible sample paths Ω

2-Period-Binomial-Model:

3. path ω_3:



Asset price dynamics in "path space model":

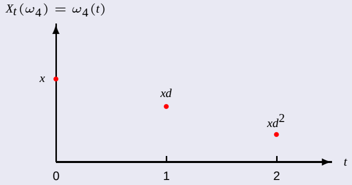
- observation time $T = n$
- time set $I = \{0, 1, 2, \ldots, n\}$ (in units)
- capital growth factor $Y_i = u_i$ or d_i, with $0 < d_i < e^r < u_i$
- asset price at time $t \in I$:

$$X_0 := x, \qquad X_t = x \cdot \prod_{i=1}^{t} Y_i$$

- random paths $\omega = (X_t : t \in I)$
- canonical space of all possible sample paths Ω

2-Period-Binomial-Model:

4. path ω_4:



Probabilistic dynamics:

- two possible models:

$$(\mathcal{H}) \quad Q^{\mathcal{H}}[Y_i = u_i] = q_i^{\mathcal{H}} = 1 - Q^{\mathcal{H}}[Y_i = d_i] \quad \text{with} \quad q_i^{\mathcal{H}} \in (0,1)$$
$$(\mathcal{A}) \quad Q^{\mathcal{A}}[Y_i = u_i] = q_i^{\mathcal{A}} = 1 - Q^{\mathcal{A}}[Y_i = d_i] \quad \text{with} \quad q_i^{\mathcal{A}} \in (0,1)$$

- assumption: $q_i^{\mathcal{H}} \neq q_i^{\mathcal{A}}$ for at least one $i \in \{1, \dots, n\}$

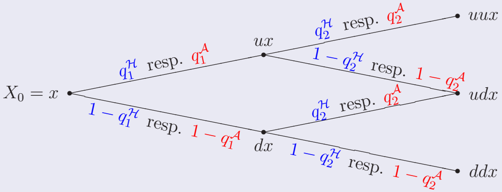
Probabilistic dynamics:

- two possible models:

$$(\mathcal{H})\quad Q^{\mathcal{H}}[Y_i = u_i] = q_i^{\mathcal{H}} = 1 - Q^{\mathcal{H}}[Y_i = d_i] \quad \text{with} \quad q_i^{\mathcal{H}} \in (0,1)$$
$$(\mathcal{A})\quad Q^{\mathcal{A}}[Y_i = u_i] = q_i^{\mathcal{A}} = 1 - Q^{\mathcal{A}}[Y_i = d_i] \quad \text{with} \quad q_i^{\mathcal{A}} \in (0,1)$$

- assumption: $q_i^{\mathcal{H}} \neq q_i^{\mathcal{A}}$ for at least one $i \in \{1, \ldots, n\}$

Bayes decisions



$Q^{\mathcal{H}}$, $q_i^{\mathcal{H}}$ vs. $Q^{\mathcal{A}}$, $q_i^{\mathcal{A}}$

$\Pr[\mathcal{H}] = pr_{\mathcal{H}}^{prior} = 0.5$ $\qquad$ $\Pr[\mathcal{A}] = pr_{\mathcal{A}}^{prior} = 0.5$

Example 1

- total amount of money $m = 10000$ Euro
- $T = 1$ unit $= \frac{1}{4}$ year
- **savings deposit** with annual interest rate $r = 0.19\%$
- **stock** with $\sigma = 0.1$ per year, $u = e^{\sigma\sqrt{\frac{T}{1}}} \approx 1.05$, $d = \frac{1}{u} \approx 0.95$

$$\mathcal{H}: \; q^{\mathcal{H}} = \frac{1}{2} + \frac{1}{2}\frac{1}{\sigma}\left(c_{\mathcal{H}} - \frac{\sigma^2}{2}\right)\sqrt{\frac{T}{1}} \approx \underline{0.52} \text{ with } c_{\mathcal{H}} = 0.013$$

$$c_{\mathcal{H}} \widehat{=} \text{"average growth rate" (per year)} \ldots \textbf{optimistic: } 1.3\%$$

$$\mathcal{A}: \; q^{\mathcal{A}} = \frac{1}{2} + \frac{1}{2}\frac{1}{\sigma}\left(c_{\mathcal{A}} - \frac{\sigma^2}{2}\right)\sqrt{\frac{T}{1}} \approx \underline{0.49} \text{ with } c_{\mathcal{A}} = 0.001$$

$$c_{\mathcal{A}} \widehat{=} \text{"average growth rate" (per year)} \ldots \textbf{pessimistic: } 0.1\%$$

We would either like to invest **all** the money in the stock (take the decision $d_{\mathcal{H}}$) or **all** the money in the savings deposit (take the decision $d_{\mathcal{A}}$) for the next period $(\frac{1}{4}, \frac{1}{2}]$.

Example 1

decision losses (rely on **expected wealth for** $t = 2$ **unit** $= \frac{1}{2}$ **year**):

- savings deposit: $m \cdot e^{0.0019 \cdot (\frac{1}{2} - \frac{1}{4})} = 10004.75$ Euro
- stock under model $\mathcal{H}$:

$$\begin{aligned} EQ^{\mathcal{H}}\left[\frac{m}{X_1} \cdot X_2\right] &= m\left\{q^{\mathcal{H}} \cdot u + (1 - q^{\mathcal{H}}) \cdot d\right\} \\ &= 10000 \cdot \{\underline{0.52} \cdot 1.05 + (1 - \underline{0.52}) \cdot 0.95\} = 10020 \text{ Euro} \end{aligned}$$

- stock under model $\mathcal{A}$:

$$\begin{aligned} EQ^{\mathcal{A}}\left[\frac{m}{X_1} \cdot X_2\right] &= m\left\{q^{\mathcal{A}} \cdot u + (1 - q^{\mathcal{A}}) \cdot d\right\} \\ &= 10000 \cdot \{\underline{0.49} \cdot 1.05 + (1 - \underline{0.49}) \cdot 0.95\} = 9990 \text{ Euro} \end{aligned}$$

- $d_{\mathcal{H}} \widehat{=}$ **invest all the money in the stock**

$$\begin{aligned} \widetilde{L}_{\mathcal{H}} &:= L(d_{\mathcal{H}}, \mathcal{H}) = 0 \text{ (relatively seen)} \\ L_{\mathcal{A}} &:= L(d_{\mathcal{H}}, \mathcal{A}) = 10004.75 - 9990 = 14.75 \text{ Euro} \end{aligned}$$

- $d_{\mathcal{A}} \widehat{=}$ **invest all the money in the savings deposit**

$$\begin{aligned} \widetilde{L}_{\mathcal{A}} &:= L(d_{\mathcal{A}}, \mathcal{A}) = 0 \text{ (relatively seen)} \\ L_{\mathcal{H}} &:= L(d_{\mathcal{A}}, \mathcal{H}) = 10020 - 10004.75 = 15.25 \text{ Euro} \end{aligned}$$

Outline

density of $Q^{\mathcal{A}}$ with respect to $Q^{\mathcal{H}}$

$$Z_T(\omega) = \begin{cases} \frac{q_1^{\mathcal{A}} \cdots\cdots q_n^{\mathcal{A}}}{q_1^{\mathcal{H}} \cdots\cdots q_n^{\mathcal{H}}}, & \text{if } X_T(\omega) = x\, u_1 \ldots u_n, \\ \frac{q_1^{\mathcal{A}} \cdots\cdots q_{n-1}^{\mathcal{A}}\,(1-q_n^{\mathcal{A}})}{q_1^{\mathcal{H}} \cdots\cdots q_{n-1}^{\mathcal{H}}\,(1-q_n^{\mathcal{H}})}, & \text{if } X_T(\omega) = x\, u_1 \ldots u_{n-1}\, d_n, \\ \frac{q_1^{\mathcal{A}} \ldots q_{n-2}^{\mathcal{A}}\,(1-q_{n-1}^{\mathcal{A}})\, q_n^{\mathcal{A}}}{q_1^{\mathcal{H}} \ldots q_{n-2}^{\mathcal{H}}\,(1-q_{n-1}^{\mathcal{H}})\, q_n^{\mathcal{H}}}, & \text{if } X_T(\omega) = x\, u_1 \ldots u_{n-2}\, d_{n-1}\, u_n, \\ \frac{q_1^{\mathcal{A}} \ldots q_{n-2}^{\mathcal{A}}\,(1-q_{n-1}^{\mathcal{A}})\,(1-q_n^{\mathcal{A}})}{q_1^{\mathcal{H}} \ldots q_{n-2}^{\mathcal{H}}\,(1-q_{n-1}^{\mathcal{H}})\,(1-q_n^{\mathcal{H}})}, & \text{if } X_T(\omega) = x\, u_1 \ldots u_{n-2}\, d_{n-1}\, d_n, \\ \vdots & \vdots \\ \vdots & \vdots \\ \frac{(1-q_1^{\mathcal{A}}) \ldots (1-q_n^{\mathcal{A}})}{(1-q_1^{\mathcal{H}}) \ldots (1-q_n^{\mathcal{H}})}, & \text{if } X_T(\omega) = x\, d_1 \ldots d_n \end{cases}$$

Remark

If $q_i^{\mathcal{H}} \equiv q^{\mathcal{H}}$ and $q_i^{\mathcal{A}} \equiv q^{\mathcal{A}}$, then Z_T depends only on X_T (and not on the path to X_T).

Hellinger integral

For $\alpha \in \mathbb{R}$:

$$H_\alpha(Q^{\mathcal{A}} \| Q^{\mathcal{H}}) = \int \{g^{\mathcal{A}}\}^\alpha \{g^{\mathcal{H}}\}^{1-\alpha} \, d\mu = EQ^{\mathcal{H}}\left[(Z_T)^\alpha \right]$$

$$\begin{aligned}
= \; & (q_1^{\mathcal{A}} \dots q_n^{\mathcal{A}})^\alpha \cdot (q_1^{\mathcal{H}} \dots q_n^{\mathcal{H}})^{1-\alpha} \\
& + (q_1^{\mathcal{A}} \dots q_{n-1}^{\mathcal{A}} (1 - q_n^{\mathcal{A}}))^\alpha \cdot (q_1^{\mathcal{H}} \dots q_{n-1}^{\mathcal{H}} (1 - q_n^{\mathcal{H}}))^{1-\alpha} \\
& + (q_1^{\mathcal{A}} \dots q_{n-2}^{\mathcal{A}} (1 - q_{n-1}^{\mathcal{A}}) \, q_n^{\mathcal{A}})^\alpha \cdot (q_1^{\mathcal{H}} \dots q_{n-2}^{\mathcal{H}} (1 - q_{n-1}^{\mathcal{H}}) \, q_n^{\mathcal{H}})^{1-\alpha} \\
& + (q_1^{\mathcal{A}} \dots q_{n-2}^{\mathcal{A}} (1 - q_{n-1}^{\mathcal{A}}) (1 - q_n^{\mathcal{A}}))^\alpha \cdot (q_1^{\mathcal{H}} \dots q_{n-2}^{\mathcal{H}} (1 - q_{n-1}^{\mathcal{H}}) (1 - q_n^{\mathcal{H}}))^{1-\alpha} \\
& \qquad \vdots \qquad\qquad \vdots \\
& + ((1 - q_1^{\mathcal{A}}) \dots (1 - q_n^{\mathcal{A}}))^\alpha \cdot ((1 - q_1^{\mathcal{H}}) \dots (1 - q_n^{\mathcal{H}}))^{1-\alpha} ,
\end{aligned}$$

where $g^{\mathcal{A}} = \frac{dQ^{\mathcal{A}}}{d\mu}\Big|_I$ and $g^{\mathcal{H}} = \frac{dQ^{\mathcal{H}}}{d\mu}\Big|_I$ are the densities with respect to the specially chosen reference law $\mu = Q^{\mathcal{H}}$ (n time points).

I_α-divergence

For $\alpha \in \mathbb{R}$:

$$I_\alpha(Q^{\mathcal{A}} \| Q^{\mathcal{H}}) = \int f_\alpha \left(\frac{dQ^{\mathcal{A}}}{dQ^{\mathcal{H}}} \Big|_I \right) dQ^{\mathcal{H}} ,$$

with the nonnegative functions $f_\alpha : [0, \infty) \to [0, \infty)$ defined by

$$f_\alpha(\rho) = \begin{cases} -\log \rho + \rho - 1, & \text{if } \alpha = 0, \\ \frac{\alpha\rho + 1 - \alpha - \rho^\alpha}{\alpha(1-\alpha)}, & \text{if } \alpha \in \mathbb{R} \backslash \{0, 1\}, \\ \rho \log \rho + 1 - \rho, & \text{if } \alpha = 1. \end{cases}$$

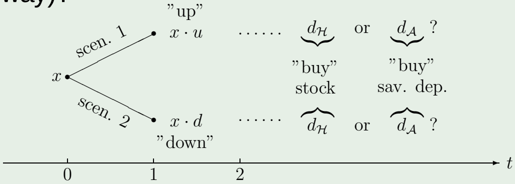
For $\alpha \in \mathbb{R} \setminus \{0, 1\}$ that holds

$$I_\alpha(Q^{\mathcal{A}} \| Q^{\mathcal{H}}) = \frac{1 - H_\alpha(Q^{\mathcal{A}} \| Q^{\mathcal{H}})}{\alpha(1-\alpha)} .$$

Outline

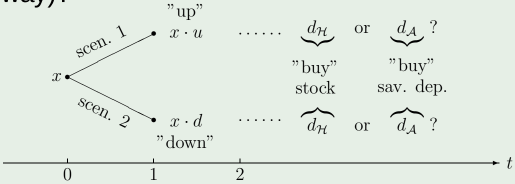
Example 1 (continuing)

How to plan (now) an optimal decision way/decision rule (in an Bayesianway)?



Example 1 (continuing)

How to plan (now) an optimal decision way/decision rule (in an Bayesianway)?



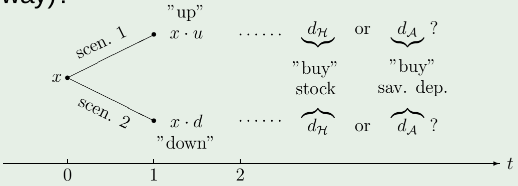
Mean loss

- decision rule $\delta : \Omega \longrightarrow \{d_{\mathcal{H}}, d_{\mathcal{A}}\}, \; \delta(\omega) = \begin{cases} d_{\mathcal{A}}, & \text{if} \quad \omega \in G, \\ d_{\mathcal{H}}, & \text{if} \quad \omega \notin G, \end{cases}$
- mean loss:

$$\begin{aligned} \mathcal{L}(\delta) &= \mathcal{L}_x(\delta_G) \\ &= L_{\mathcal{H}} \; \Pr[\delta_G(\omega) = d_{\mathcal{A}}, \mathcal{H}] + L_{\mathcal{A}} \; \Pr[\delta_G(\omega) = d_{\mathcal{H}}, \mathcal{A}] \\ &= L_{\mathcal{H}} \; \Pr[\omega \in G | \mathcal{H}] \cdot \Pr[\mathcal{H}] + L_{\mathcal{A}} \; \Pr[\omega \in \Omega \setminus G | \mathcal{A}] \cdot \Pr[\mathcal{A}] \\ &= L_{\mathcal{H}} \, pr_{\mathcal{H}}^{prior} \; Q^{\mathcal{H}}[G] + L_{\mathcal{A}} \, pr_{\mathcal{A}}^{prior} \; Q^{\mathcal{A}}[\Omega \setminus G] \end{aligned}$$

Example 1 (continuing)

How to plan (now) an optimal decision way/decision rule (in an Bayesianway)?



Mean loss

- decision rule $\delta : \Omega \longrightarrow \{d_{\mathcal{H}}, d_{\mathcal{A}}\}$, $\delta(\omega) = \begin{cases} d_{\mathcal{A}}, & \text{if} \quad \omega \in G, \\ d_{\mathcal{H}}, & \text{if} \quad \omega \notin G, \end{cases}$
- mean loss:

$$\begin{aligned} \mathcal{L}(\delta) &= \mathcal{L}_x(\delta_G) \\ &= L_{\mathcal{H}} \ \Pr[\delta_G(\omega) = d_{\mathcal{A}}, \mathcal{H}] + L_{\mathcal{A}} \ \Pr[\delta_G(\omega) = d_{\mathcal{H}}, \mathcal{A}] \\ &= L_{\mathcal{H}} \ \Pr[\omega \in G|\mathcal{H}] \cdot \Pr[\mathcal{H}] + L_{\mathcal{A}} \ \Pr[\omega \in \Omega \setminus G|\mathcal{A}] \cdot \Pr[\mathcal{A}] \\ &= L_{\mathcal{H}} \, pr_{\mathcal{H}}^{prior} \ Q^{\mathcal{H}}[G] + L_{\mathcal{A}} \, pr_{\mathcal{A}}^{prior} \ Q^{\mathcal{A}}[\Omega \setminus G] \qquad \longrightarrow \text{min!} \end{aligned}$$

1st decision method

- optimal decision rule:

$$\delta_{opt}(\omega) = \delta_{G_{\min}} = \begin{cases} d_{\mathcal{A}}, & \text{if} \quad \omega \in G_{\min}, \\ d_{\mathcal{H}}, & \text{if} \quad \omega \notin G_{\min}, \end{cases}$$

  with "optimal decision set" $G_{\min} := \left\{ \text{paths } \omega \in \Omega : \; Z_T(\omega) \geq \frac{L_{\mathcal{H}}\, pr_{\mathcal{H}}^{prior}}{L_{\mathcal{A}}\, pr_{\mathcal{A}}^{prior}} \right\}$
- Bayes risk (minimal mean loss):

$$\mathcal{L}(\delta_{opt}) = \mathcal{L}(\delta_{G_{\min}}) = \cdots = EQ^{\mathcal{H}} \left[\min\{L_{\mathcal{H}}\, pr_{\mathcal{H}}^{prior}, L_{\mathcal{A}}\, pr_{\mathcal{A}}^{prior}\, Z_T\} \right]$$

1st decision method

- optimal decision rule:

$$\delta_{\text{opt}}(\omega) = \delta_{G_{\min}} = \begin{cases} d_{\mathcal{A}}, & \text{if} \quad \omega \in G_{\min}, \\ d_{\mathcal{H}}, & \text{if} \quad \omega \notin G_{\min}, \end{cases}$$

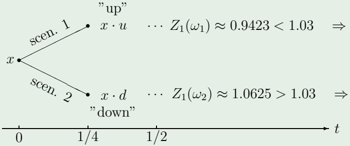
with "optimal decision set" $G_{\min} := \left\{ \text{paths } \omega \in \Omega : Z_T(\omega) \geq \frac{L_{\mathcal{H}}\, pr_{\mathcal{H}}^{prior}}{L_{\mathcal{A}}\, pr_{\mathcal{A}}^{prior}} \right\}$

- Bayes risk (minimal mean loss):

$$\mathcal{L}(\delta_{\text{opt}}) = \mathcal{L}(\delta_{G_{\min}}) = \cdots = EQ^{\mathcal{H}} \left[\min\{ L_{\mathcal{H}}\, pr_{\mathcal{H}}^{prior}, L_{\mathcal{A}}\, pr_{\mathcal{A}}^{prior}\, Z_T \} \right]$$

Example 1 (continuing)

$$G_{\min} = \left\{ \omega \in \Omega : Z_1 \geq \frac{0.5 \cdot 15.25}{0.5 \cdot 14.75} \approx 1.03 \right\}$$



$\cdots\ Z_1(\omega_1) \approx 0.9423 < 1.03 \quad \Rightarrow \quad$ take decision $d_{\mathcal{H}}$, i.e. "buy" stock

$\cdots\ Z_1(\omega_2) \approx 1.0625 > 1.03 \quad \Rightarrow \quad$ decision $d_{\mathcal{A}}$, i.e. "buy" savings deposit

$\mathcal{L}(\delta_{G_{\min}}) = 0.4850$ Euro (of total investment of 10000 Euro)

2nd decision method (⇔ 1st method)

- decision criterium:

$$\begin{aligned}
&\text{If} \quad pr_{\mathcal{H}}^{post} L_{\mathcal{H}} \leq pr_{\mathcal{A}}^{post} L_{\mathcal{A}}, \quad \text{then decide for } d_{\mathcal{A}}, \\
&\text{If} \quad pr_{\mathcal{H}}^{post} L_{\mathcal{H}} > pr_{\mathcal{A}}^{post} L_{\mathcal{A}}, \quad \text{then decide for } d_{\mathcal{H}},
\end{aligned}$$

  where $pr_{\mathcal{H}}^{post} = \frac{pr_{\mathcal{H}}^{prior}}{pr_{\mathcal{H}}^{prior} + (1 - pr_{\mathcal{H}}^{prior})\ Z_T}$, $pr_{\mathcal{A}}^{post} = \frac{(1 - pr_{\mathcal{H}}^{prior})\ Z_T}{pr_{\mathcal{H}}^{prior} + (1 - pr_{\mathcal{H}}^{prior})\ Z_T}$
  are the "overall evidence" probabilities posterior to our observation between the times 0 and T.
- Bayes risk as with 1st decision method
- Bayes factor: $\mathcal{B}_T = \frac{\text{posterior odds ratio of } \mathcal{A} \text{ to } \mathcal{H}}{\text{prior odds ratio of } \mathcal{A} \text{ to } \mathcal{H}} = Z_T$

2nd decision method ($\Leftrightarrow$ 1st method)

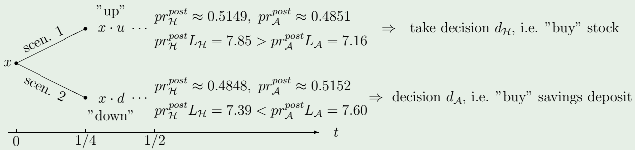
- decision criterium:

$$\text{If} \quad pr_{\mathcal{H}}^{post} L_{\mathcal{H}} \leq pr_{\mathcal{A}}^{post} L_{\mathcal{A}}, \quad \text{then decide for } d_{\mathcal{A}},$$
$$\text{If} \quad pr_{\mathcal{H}}^{post} L_{\mathcal{H}} > pr_{\mathcal{A}}^{post} L_{\mathcal{A}}, \quad \text{then decide for } d_{\mathcal{H}},$$

where $pr_{\mathcal{H}}^{post} = \frac{pr_{\mathcal{H}}^{prior}}{pr_{\mathcal{H}}^{prior} + (1 - pr_{\mathcal{H}}^{prior})\, Z_T}$, $pr_{\mathcal{A}}^{post} = \frac{(1 - pr_{\mathcal{H}}^{prior})\, Z_T}{pr_{\mathcal{H}}^{prior} + (1 - pr_{\mathcal{H}}^{prior})\, Z_T}$
are the "overall evidence" probabilities posterior to our observation between the times 0 and T.

- Bayes risk as with 1st decision method
- Bayes factor: $\mathcal{B}_T = \frac{\text{posterior odds ratio of } \mathcal{A} \text{ to } \mathcal{H}}{\text{prior odds ratio of } \mathcal{A} \text{ to } \mathcal{H}} = Z_T$

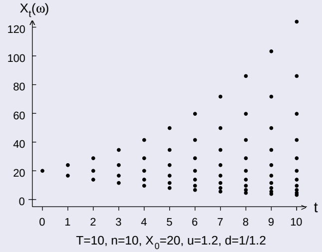
Example 1 (continuing)



$\mathcal{L}(\delta_{G_{\min}}) = 0.4850$ Euro (of total investment of 10000 Euro)

Outline

Special homogeneous case (SPH) with n periods



observations at times $0, \frac{T}{n}, 2\frac{T}{n}, \ldots, n\frac{T}{n} = T$, with fixed **approximation step** n

in **each knot** like in the one-period-model, independently of past/history, with

$$u_i = u_{(n)} := e^{\sigma\sqrt{\frac{T}{n}}}, \qquad d_i = d_{(n)} := e^{-\sigma\sqrt{\frac{T}{n}}},$$

$$\textbf{for } \mathcal{H}\textbf{:} \quad q_i^{\mathcal{H}} = q_{(n)}^{\mathcal{H}} := \frac{1}{2} + \frac{1}{2}\frac{1}{\sigma}\left(c_{\mathcal{H}} - \frac{\sigma^2}{2}\right)\sqrt{\frac{T}{n}} \quad \ldots\ldots \quad Q^{\mathcal{H}}$$

$$\textbf{for } \mathcal{A}\textbf{:} \quad q_i^{\mathcal{A}} = q_{(n)}^{\mathcal{A}} := \frac{1}{2} + \frac{1}{2}\frac{1}{\sigma}\left(c_{\mathcal{A}} - \frac{\sigma^2}{2}\right)\sqrt{\frac{T}{n}} \quad \ldots\ldots \quad Q^{\mathcal{A}}$$

with volatility $\sigma > 0$, and "average growth rate" $c_{\mathcal{H}} \in \mathbb{R}$ resp. $c_{\mathcal{A}} \in \mathbb{R}$

homogeneous binomial models $\begin{matrix} q_{(n)}^{\mathcal{H}} \\ q_{(n)}^{\mathcal{A}} \end{matrix} \xrightarrow{n \longrightarrow \infty}$ **geom. Brown. motions** $\begin{matrix} (c_{\mathcal{H}}, \sigma^2) \\ (c_{\mathcal{A}}, \sigma^2) \end{matrix}$

Some Results of Convergence

- Hellinger integral:

$$H_{\alpha}^{(n)}\left(Q^{\mathcal{A}} \| Q^{\mathcal{H}}\right) \stackrel{n\to\infty}{\longrightarrow} \exp\left\{\left(\frac{c_{\mathcal{A}} - c_{\mathcal{H}}}{\sigma}\right)^2 T\,\frac{\alpha(\alpha-1)}{2}\right\} =: C_{\alpha}$$

- I_{α}-divergence:
  - $\alpha \in \mathbb{R} \setminus \{0,1\}: \quad I_{\alpha}^{(n)}(Q^{\mathcal{A}} \| Q^{\mathcal{H}}) \stackrel{n\to\infty}{\longrightarrow} \frac{1-C_{\alpha}}{\alpha(1-\alpha)}$
  - $\alpha = 1: \quad I_{1}^{(n)}(Q^{\mathcal{A}} \| Q^{\mathcal{H}}) \stackrel{n\to\infty}{\longrightarrow} \frac{1}{2}\left(\frac{c_{\mathcal{A}}-c_{\mathcal{H}}}{\sigma}\right)^2 T$ (relative entropy)
  - $\alpha = 0: I_{0}^{(n)}(Q^{\mathcal{A}} \| Q^{\mathcal{H}}) \stackrel{n\to\infty}{\longrightarrow} \frac{1}{2}\left(\frac{c_{\mathcal{A}}-c_{\mathcal{H}}}{\sigma}\right)^2 T$
- minimal mean loss:

$$\mathcal{L}(\delta_{G_{\min}}^{T,n}) \stackrel{n\to\infty}{\longrightarrow} \lambda_{\mathcal{A}} \cdot \left(1 - \Phi_{\mathcal{N}(0,1)}(\theta_1)\right) + \lambda_{\mathcal{H}} \cdot \Phi_{\mathcal{N}(0,1)}(\theta_2)$$

with: $\theta_1 := \frac{\Delta\sqrt{T}}{2} - \frac{\ln(\lambda_{\mathcal{H}}/\lambda_{\mathcal{A}})}{\Delta\sqrt{T}}$, $\theta_2 := -\frac{\Delta\sqrt{T}}{2} - \frac{\ln(\lambda_{\mathcal{H}}/\lambda_{\mathcal{A}})}{\Delta\sqrt{T}}$, and $\Delta := \frac{\mu_{\mathcal{H}}-\mu_{\mathcal{A}}}{\sigma}$, $\lambda_{\mathcal{H}} := pr_{\mathcal{H}}^{prior} L_{\mathcal{H}}$, $\lambda_{\mathcal{A}} := pr_{\mathcal{A}}^{prior} L_{\mathcal{A}}$

(conform with Stummer/Vajda 2005)

- further investigations (e.g. more general losses......)

References:

J.C. Cox, S.A. Ross and M. Rubinstein.
Option Pricing: A Simplified Approach.
Journal of Financial Economics, 7:229–263, 1979.

W. Lao and W. Stummer.
Limits of Divergences and Bayes Risks for Some Time-Discrete Asset Value Model Choices.
Preprint, 2006.

F. Liese and I. Vajda.
Convex Statistical Distances.
Teubner, Leipzig, 1987.

T.R.C. Read and N.A.C. Cressie.
Goodness-of-Fit Statistics for Discrete Multivariate Data.
Springer, New York, 1988.

W. Stummer and I. Vajda.
Optimal Statistical Decisions About Some Alternative Financial Models.
Journal of Econometrics, to appear, 2006.