Variational Inequality and Complementarity Problems

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Summary of Talk

- Source Problems
  - Optimization Problem
  - Free Boundary Problem
  - Equilibrium Problem
  - ......
- Algorithms
  - Josephy-Newton-Type Algorithm
  - Nonsmooth Newton Algorithms
  - Smoothing Algorithms
  - ......

The variational inequality problem

Given $\Omega \subseteq R^n$ nonempty, and $F : R^n \rightarrow R^n$. The variational inequality problem, denoted by $VIP(\Omega, F)$, is to find a vector $x^* \in \Omega$ such that

$$(y - x^*)^T F(x^*) \geq 0, \qquad \forall y \in \Omega.$$

The nonlinear complementarity problem

Given $F : R^n \rightarrow R^n$. The nonlinear complementarity problem, denoted by $NCP(f)$, is to find a vector x^* such that

$$x^* \geq 0, \qquad F(x^*) \geq 0, \qquad {x^*}^T F(x^*) = 0.$$

- **Source Problems**

  - **Optimization Problem**

Given $\Omega \subseteq R^n$ nonempty, and $f : R^n \to R$. A minimization problem is to find a vector $x^* \in \Omega$ such that for any $x \in \Omega$ (or $x \in \Omega \cap \mathcal{N}(x^*)$) it holds $f(x^*) \leq f(x)$, where $\mathcal{N}(x^*)$ denotes a certain neighborhood of x^*. We call x^* a global (or local) minimizer of function f, and call $f(x^*)$ a global (or local) minimum of f.

Notation:

$$\min f(x)$$

$$S.t.\ x \in \Omega.$$

Optimization Problem and VIP

Let $\Omega \subseteq R^n$ be nonempty and convex, let $f : R^n \rightarrow R$ be differentiable and $F(x) = \nabla f(x)$.

- x^* solves $VIP(\Omega, F)$ if and only if there is no feasible decent direction at x^*.

- Furthermore, if f is pseudo-convex, then any solution of $VIP(\Omega, F)$ is a global minimizer of f over Ω.

- **Source Problems**

  - **Free Boundary Problem**

Find a function $u(t,s) : D \subseteq R^2 \to R$ such that

$$\begin{cases} \triangle u &= & \varphi(t,s,u,u'_t,u'_s) & \text{in} & D_0 \\ \triangle u &\leq & \varphi(t,s,u,u'_t,u'_s) & \text{in} & D_- \\ \triangle u &\geq & \varphi(t,s,u,u'_t,u'_s) & \text{in} & D_+ \\ u &= & \psi(t,s) & \text{on} & \partial D, \end{cases}$$

where the domains

$$\begin{aligned} D_0 &:= \{(t,s) \in D : \underline{u} < u < \overline{u}\} \\ D_- &:= \{(t,s) \in D : u = \underline{u}\} \\ D_+ &:= \{(t,s) \in D : u = \overline{u}\} \end{aligned}$$

are unknown.

Free Boundary Problem and VIP $-$ I

Let the free boundary problem have a unique solution u. How to approximate it ?

Impose square mesh

$$(t_m, s_l) = (mh, lh), \quad \text{where} \begin{cases} h = \frac{1}{k+1} \\ m, l = 1, \ldots, k. \end{cases}$$

Find $u_{m,l}$, $m, l = 1, \ldots, k$ such that

$$u(t_m, s_l) \simeq u_{m,l}.$$

Free Boundary Problem and VIP – II

- Approximate $\triangle u$ at (t_m, s_l) by:

$$\frac{u_{m-1,l} + u_{m+1,l} + u_{m,l-1} + u_{m,l+1} - 4u_{m,l}}{h^2}.$$

- Approximate u'_t at (t_m, s_l) by:

$$\frac{u_{m+1,l} - u_{m-1,l}}{2h}.$$

- Approximate u'_s at (t_m, s_l) by:

$$\frac{u_{m,l+1} - u_{m,l-1}}{2h}.$$

Free Boundary Problem and VIP – III

$u = (u_{m,l}) \in R^{k^2}$ is a solution of $VIP(\Omega, F)$,

$$F_{m,l} := -u_{m-1,l} - u_{m+1,l} - u_{m,l-1} - u_{m,l+1} + 4u_{m,l}$$

$$+h^2\varphi(t_m, s_l, u_{m,l}, \tfrac{u_{m+1,l}-u_{m-1,l}}{2h}, \tfrac{u_{m,l+1}-u_{m,l-1}}{2h})$$

$$\Omega := \{u = (u_{m,l}) | (\underline{u}(t_m, s_l)) \leq u \leq (\overline{u}(t_m, s_l))\}$$

- **Source Problems**
  - **Equilibrium Problem**

Set of players: $N = \{1, 2, \ldots, n\}$,

Strategy vector: $x_i \in R^{m_i}$,

Strategy set: $X_i \subseteq R^{m_i}$,

Strategy space: $X = \prod_{i \in N} X_i$,

Utility function: $u_i : X \to R$,

Utility vector: $u = (u_1, u_2, \ldots, u_n)^T$.

Nash Equilibrium and VIP

A Nash equilibrium $x^* \in X$ of the game $NE(X, u)$ is defined as a point at which no player can unilaterally increase his utility, that is:

$$u_i(x_i^*, x_{N-\{i\}}^*) \geq u_i(x_i, x_{N-\{i\}}^*).$$

Let X_i be nonempty, closed and convex subset, and $u_i : X \to R$ be once continuously differentiable and pseudo-concave w.r.t. x_i for all $i \in N$. Then x^* is a Nash equilibrium if and only if $x^* \in X$ is a solution of $VIP(X, F)$, where

$$F(x) = (F_i(x) : i \in N), \quad F_i(x) = -\partial u_i(x)/\partial x_i.$$

- **Algorithms**

  - **Josephy-Newton-Type Algorithm – I**

General Iterative Scheme:

Generate $\{x^k\}_{k=0}^{\infty} \subset R^n$ such that

$$x^{k+1} \quad \text{solves} \quad VIP(\Omega, F^k)$$

where

$$F^k(x) = F(x^k) + A(x^k)(x - x^k).$$

- **Algorithms**

  - **Josephy-Newton-Type Algorithm – II**

$D(x)$:	diagonal part of $F'(x)$
$L(x)$:	lower triangular part of $F'(x)$
$U(x)$:	upper triangular part of $F'(x)$
ω^* :	a scalar parameter $\in (0, 2)$
G :	a fixed, symmetric, positive definite matrix

- **Algorithms**

  - **Josephy-Newton-Type Algorithm – III**

Frequently Used Algorithms:

(a) Josephy-Newton Algorithm

$$A(x^k) = F'(x^k)$$

(b) Quasi-Newton Algorithm

$$A(x^k) \simeq F'(x^k)$$

(c) Projection Algorithm

$$A(x^k) = G$$

- **Algorithms**

  – **Josephy-Newton-Type Algorithm – IV**

Other Algorithms:

(d) Successive Overrelaxation Algorithm (SOR)

$$A(x^k) = \begin{cases} L(x^k) + D(x^k)/\omega^* \\ U(x^k) + D(x^k)/\omega^* \end{cases}$$

(e) Symmetrized Newton Algorithm

$$A(x^k) = \frac{1}{2}(F'(x^k) + F'(x^k)^T)$$

. . . . . .

- **Algorithms**

  – **Josephy-Newton-Type Algorithm – V**

Merit Function

Function $\theta : R^n \rightarrow R_+$ is a merit function provided that $\theta(x^*) = 0$ if and only if x^* solves $VIP(\Omega, F)$.

Linear Search

Given x^k. Compute $\tilde{x}^{k+1}$ such that

$$\tilde{x}^{k+1} \quad \text{solves} \quad VIP(\Omega, F^k).$$

Set $d^k := \tilde{x}^{k+1} - x^k$. Find λ^k such that

$$\theta(x^k + \lambda^k d^k) = \min_{\lambda \in (0, \bar{\lambda})} \theta(x^k + \lambda d^k).$$

Set

$$x^{k+1} := x^k + \lambda^k d^k.$$

Equation Reformulation

Find $H : R^n \rightarrow R^n$ such that x^* solves $VIP(\Omega, F)$ if and only if

$$H(x) = 0$$

$$\uparrow$$

Usually Nonsmooth

- $H(x) = x - Pr_{\Omega,G}(x - G^{-1}F(x))$

- $H(x) = \text{mid}\{x - L, x - U, F(x)\}$, when

$$\Omega = \{x \in R^n | L \leq x \leq U\}$$

- $H(x) = \min\{x, F(x)\} = 0$ when $\Omega = R^n_+$.

- **Algorithms**

  - **Nonsmooth Newton Algorithms**

Given $x^k \in R^n$. Generate $\{x^k\}_{k=0}^{\infty}$ such that

$$x^{k+1} := x^k - A(x^k)^{-1} H(x^k),$$

where

$$A(x^k) \in \partial H(x^k).$$

$$\uparrow$$

Generalized Jacobian

$$\boxed{\partial H(x) := \overline{co}\{J = \lim_{m \to \infty} G'(y^m)\}}$$

where $y^m \to x$, H is differentiable at each y^m.

Homotopy Reformulation

Find $H(s,x) : R \times R^n \rightarrow R^n$ such that x^* solves $VIP(\Omega, F)$ if and only if

$$\begin{array}{c} H(0,x) = 0 \\ \uparrow \\ \text{Usually Nonsmooth} \end{array}$$

$H(s,x)$ differentiable when $s \neq 0$.

Example for $NCP(F)$

$$H(s,x) = (\sqrt{x_i^2 + F_i(x)^2 + s} - (x_i + F_i(x)))$$

- **Algorithms**

  – **Smoothing Algorithms**

Choose $\{s_\kappa\} \subseteq R_{++}$ such that $s_\kappa \downarrow 0$. Compute an approximate solution x^{κ,n_κ} of

$$H(s_\kappa, x) = 0.$$

Starting from x^{κ,n_κ}, an approximate solution $x^{\kappa+1,n_{\kappa+1}}$ of

$$H(s_{\kappa+1}, x) = 0.$$

Obtain a sequence $\{x^{\kappa,n_\kappa}\}_{\kappa=0}^{\infty} \to x^*$.

Software

- PATH
  ftp://ftp.cs.wisc.edu/math-prog
  /solvers/path/matlab/

- LANCELOT

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Literature

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Thank You All !