Wavelet collocation method to Partial Differential Equations

Haojun Li
li@math.uni-karlsruhe.de

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Department of Mathematics

Outline

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2. Numerical example
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Wavelet collocation method

Definition

A sequence $\{V_j\}_{j\in\mathbb{Z}}$ of closed subspace of $L^2(\mathbb{R})$ is a Multi-resolution approximation if the following 6 properties are satisfied:

$$\forall (j,k)\in\mathbb{Z}^2, f(t)\in V_j \Leftrightarrow f(t-2^j k)\in V_j, \tag{1}$$

$$\forall j\in\mathbb{Z}, V_{j+1}\subset V_j, \tag{2}$$

$$\forall j\in\mathbb{Z}, f(t)\in V_j \Leftrightarrow f(\frac{t}{2})\in V_{j+1}, \tag{3}$$

$$\lim_{j\to+\infty} V_j = \bigcap_{j=-\infty}^{+\infty} V_j = \{0\}, \tag{4}$$

$$\lim_{j\to-\infty} V_j = \overline{\bigcup_{j=-\infty}^{+\infty} V_j} = L^2(\mathbb{R}) \tag{5}$$

$$\text{There exists } \phi \text{ such that } \phi(t-n)_{n\in\mathbb{Z}} \text{ is a Riesz basis of } V_0. \tag{6}$$

Wavelet collocation method

Remark

1. $\phi(t)$ *is called a* *scaling function*, ϕ *is called* *orthogonal* *if*

$$\langle \phi(t), \phi(t-k) \rangle = \delta_{0,k} \tag{7}$$

2. *The MRA is called* *orthodogonal* *if* ϕ *is orthogonal.*
3. $\phi_{n,k}(t) = 2^{-\frac{n}{2}} \phi(2^{-n}t - k)$ *is a Riesz basis of* V_n.
4. *Since* $V_0 \subset V_{-1}$, ϕ *can be written in terms of the basis of* V_{-1} *as*

$$\phi(t) = \sum_k h_k \phi_{-1,k}(t) = \sqrt{2} \sum_k h_k \phi(2t - k), \tag{8}$$

which is called *refinable equation* *and* h_k*'s are called* *recursion coefficients*.

5. $$h(\xi) = \frac{1}{\sqrt{2}} \sum_{k=k_0}^{k_1} h_k e^{-ik\xi} \tag{9}$$

is called *symbol of refinable functions*.

Wavelet collocation method

Theorem

A necessary condition for orthogonality is

$$\sum_k h_k \bar{h}_{k-2l} = \delta_{0,l} \tag{10}$$

Lemma

If ϕ is a solution of the refinable equation with compact support, then

$$supp\ \phi = [k_0, k_1]. \tag{11}$$

Theorem

The orthogonality condition in equation in (10) is equivalent to

$$|h(\xi)|^2 + |h(\xi + \pi)|^2 = 1. \tag{12}$$

Wavelet collocation method

Theorem

Let W_n be orthogonal completement of V_n in V_{n-1}, we have the following theorem,for any orthogonal MRA with scaling function ϕ,

1. $$\bigoplus_n W_n \text{ is dense in } L^2.$$
2. $$W_k \perp W_n \text{ if } k \neq n.$$
3. $$f(t) \in W_n \Leftrightarrow f(2t) \in W_{n+1} \text{ for all } n \in \mathbb{Z}.$$
4. $$f(t) \in W_n \Leftrightarrow f(t - 2^{-n}k) \in W_n \text{ for all } n, k \in \mathbb{Z}.$$
5. *There exists a function $\psi \in L^2$ so that $\{\psi(t-k) : k \in \mathbb{Z}\}$ forms an orthogonal stable basis of W_0, and $\{\psi_{n,k} : n, k \in \mathbb{Z}\}$ forms a stable basis of L^2.*
6. *Since $\psi \in V_1$, it can be represented as*
$$\psi(t) = \sqrt{2}\sum_k g_k \phi(2t-k) \tag{13}$$
for some coefficients g_k, If h_k are the recursion coefficients of ϕ, then we can choose
$$g_k = (-1)^k h_{N-k} \tag{14}$$
where N is any odd number.

The function ψ is called the wavelet function or mother wavelet. ϕ and ψ together form a wavelet.

Wavelet collocation method

Theorem

If ψ is a wavelet with p vanishing moments that generates an orthonormal basis of $L^2(\mathbb{R})$, then it has a support of size larger than or equal to $2p-1$. A Daubechies wavelet has a minimum size support equal to $[-p+1, p]$. The support of the corresponding scaling function ϕ is $[0, 2p-1]$.

Wavelet collocation method

Deslauries-Dubuc interpolating function ϕ of order $2p-1$ is given by an autocorrelation function of the Daubechies compactly supported orthogonal scaling functions ϕ_0 of p vanishing wavelet moments as

$$\phi(x) = \int_{-\infty}^{+\infty} \phi_0(u)\phi_0(u-x)du \tag{15}$$

which has even symmetry and minimum support of $[-2p+1, 2p-1]$, and reproduces polynomials of order $2p-1$. We choose $\phi(x)$ as a scaling function, which satisfies the so-called dilation relation or the two-scale relation

$$\phi(x) = \sum_{k=-\infty}^{+\infty} h_k^* \phi(2x-k). \tag{16}$$

The filter coefficients h_k^* in (16) are obtained from the Daubechies filter of the compactly supported wavelets h_k by

$$h_k^* = \sum_{m=-\infty}^{+\infty} h_m h_{m-k} \tag{17}$$

It can be easily shown that $h_{-k}^* = h_k^*$.

Wavelet collocation method

Discretization of derivatives using DD_p scaling functions, let

$$\tilde{f}(x) = \sum_{i=-\infty}^{+\infty} f_{i+\frac{1}{2}} \phi(\frac{x}{\Delta x} - i - \frac{1}{2}) \tag{18}$$

differentiate both sides of (18)we get

$$\tilde{f}'(x) = \sum_{i=-\infty}^{+\infty} f_{i+\frac{1}{2}} \phi'(\frac{x}{\Delta x} - i - \frac{1}{2}) \tag{19}$$

We use $\delta(\frac{x}{\Delta x})$ to test equation (19).

$$\langle \delta(\frac{x}{\Delta x}), \tilde{f}'(x) \rangle = \langle \delta(\frac{x}{\Delta x}), \sum_{i=-\infty}^{+\infty} f_{i+\frac{1}{2}} \phi'(\frac{x}{\Delta x} - i - \frac{1}{2}) \rangle \tag{20}$$

Thus, we obtain

$$\Delta x \tilde{f}'(0) = \sum_{i=-\infty}^{+\infty} f_{i+\frac{1}{2}} \phi'(-i - \frac{1}{2}) \tag{21}$$

Wavelet collocation method

Since ϕ is compactly supported, the number of summation is finite, also because of symmetry property of ϕ, we have,

$$\begin{aligned} \Delta x \tilde{f}'(0) = \quad & (\textstyle\sum_{i=-n}^{-1} + \sum_{i=0}^{n-1}) f_{i+\frac{1}{2}} \phi'(-i - \frac{1}{2}) \\ & = \textstyle\sum_{i=0}^{n-1} (f_{i+\frac{1}{2}} - f_{-i-\frac{1}{2}}) \phi'(-i - \frac{1}{2}) \end{aligned} \tag{22}$$

for example, discretization of $\tilde{f}'(0)$ with DD_2 is

$$\begin{aligned} \tilde{f}'(0) = \quad & \frac{1.2291666667(f(\frac{1}{2}) - f(-\frac{1}{2})) - 0.0937500000(f(\frac{3}{2}) - f(-\frac{3}{2}))}{\Delta x} \\ & + \frac{0.0104166667(f(\frac{5}{2}) - f(-\frac{5}{2}))}{\Delta x} \end{aligned} \tag{23}$$

Where finite difference scheme is

$$\tilde{f}'(0) = \frac{f(\frac{1}{2}) - f(-\frac{1}{2})}{\Delta x} \tag{24}$$

Numerical example

2D structured Time Domain Maxwell's equations are solved with Wavelet collocation methods and Finite difference method.

$$\begin{cases} \dfrac{\partial H_x}{\partial t} = \dfrac{1}{\mu}\dfrac{\partial E_y}{\partial z} \\ \dfrac{\partial H_z}{\partial t} = -\dfrac{1}{\mu}\dfrac{\partial E_y}{\partial x} \\ \dfrac{\partial E_y}{\partial t} = \dfrac{1}{\varepsilon}\left(\dfrac{\partial H_x}{\partial z} - \dfrac{\partial H_z}{\partial x}\right) \end{cases} \tag{25}$$

Numerical example

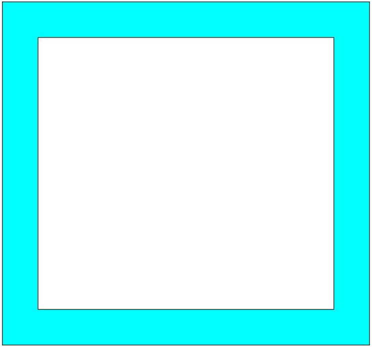

Figure: a numerical example problem

Domain size: $5.0\mu m \times 5.0\mu m$
Number of cells in each direction: 128(FDM) or 64(WCM)
Number of cells in the width of PML layer: 20(FDM) or 10(WCM)

Check four steps $t_0 = 0$, $t_1 = 4.5fs$, $t_2 = 9fs$, $t_3 = 13.5fs$

Numerical example

Wavelet collocation method: DD_2 scaling functions
Local accuracy: 3rd order (?)
Stability condition: $\Delta t \leq \frac{1}{c \sum_{i=0}^{n-1} |a(i)|} \frac{1}{\sqrt{\frac{1}{\Delta x^2} + \frac{1}{\Delta y^2} + \frac{1}{\Delta z^2}}}$
Where $a(i) = \phi'(-i - \frac{1}{2})$.

Finite Difference Method:
Local accuracy: 2nd order
Stability condition: $\Delta t \leq \frac{1}{c} \frac{1}{\sqrt{\frac{1}{\Delta x^2} + \frac{1}{\Delta y^2} + \frac{1}{\Delta z^2}}}$

Numerical example

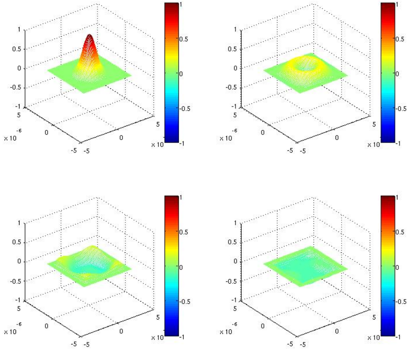

Figure: WCM

Numerical example

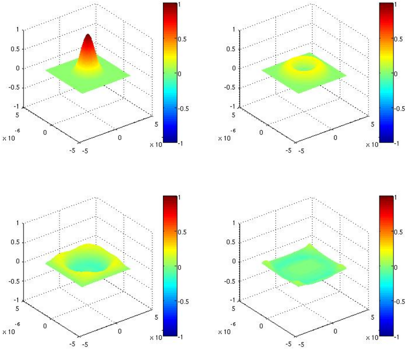

Figure: FDM

Conclusion

WCM can get more accuracy with coarser grid than FDM, thus, Δt can also be larger than that of FDM, which saves computation time effectively.

Reference

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Thank you for your attention!