Solutions to viscous model of quantum hydrodynamics

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Part I

Preliminaries

Norms depending upon a parameter $\eta \in \mathbb{C}/\{0\}$

- Let s, m be integers with $1 \leq s \leq m$, $1 \leq p < \infty$, assume the boundary $\partial\Omega$ of Ω is of class $C^{m-1,1}$ for Ω a bounded region in $\mathbb{R}^d(d = 2, 3)$, $u \in W_p^s(\Omega)$, $v \in W_p^{s-1/p}(\partial\Omega)$, define

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  1. $|||u|||_{s,p} := ||u||_{W_p^s(\Omega)} + |\eta|^{s/m}||u||_{L^p(\Omega)}$,

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  1. $|||u|||_{s,p} := ||u||_{W_p^s(\Omega)} + |\eta|^{s/m}||u||_{L^p(\Omega)}$,
  2. $|||v|||_{s-1/p,p} := ||v||_{W_p^{s-1/p}(\partial\Omega)} + |\eta|^{(s-1/p)/m}||v||_{L^p(\partial\Omega)}$.

Ellipticity with parameter in the sense of Faierman(2006)

- Let $\{s_j\}_1^N, \{m_j\}_1^N, \{r_j\}_1^N$ denote sequences of integers such that $m := s_j + m_j$ and $0 = m_1 \leq m_2 \cdots \leq m_N$, $s_1 \geq s_2 \cdots \geq s_N$.

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- Consider the boundary problem for given $f = (f_1, \cdots, f_N)$, g_j

$$\begin{cases} A(x, D)u(x) - \eta u(x) = f(x) \text{ in } \Omega, \\ B_j(x, D) = g_j(x) \text{ on } \partial\Omega \text{ for } j = 1, \cdots, mN/2, \end{cases} \tag{1}$$

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  - $A(x,D)$ is an $N \times N$ matrix operator whose entries $A_{jk}(x,D)$ are linear differential operators of order not exceeding $s_j + m_k$ and defined to be zero if $s_j + m_k < 0$;

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  - $B_j(x,D)$ is a $1 \times N$ matrix operator whose entries are linear differential operators defined on $\partial\Omega$ of order not exceeding $r_j + m_k$ and defined to be 0 if $r_j + m_k < 0$.

Ellipticity with parameter in the sense of Faierman(2006)

Let us write

$$A_{jk}(x,D)=\sum_{|\alpha|\le s_j+m_k} a_\alpha^{jk}(x)D^\alpha,$$

$$B_{jk}(x,D)=\sum_{|\alpha|\le r_j+m_k} b_\alpha^{jk}(x)D^\alpha.$$

Assumption 1 (1) $\partial\Omega$ is of class $C^{m_N+m-1,1}$;(2) for each pair $j,k,a_\alpha^{jk}\in C^{m_j-1,1(\overline{\Omega})}$ for $|\alpha|\le s_j+m_k$ if $m_j>0$, while if $m_j=0$, then $a_\alpha^{jk}\in L^\infty(\Omega)$ for $|\alpha|<s_j+m_k$ and $a_\alpha^{jk}\in C^0(\Omega)$ for $|\alpha|=s_j+m_k$; (3) for each pair $j,k,b_\alpha^{jk}\in C^{m-r_j-1,1}(\partial\Omega)$ for $|\alpha|\le r_j+m_k$.

Ellipticity with parameter in the sense of Faierman(2006)

Let $\mathcal{L}$ be a closed sector in the complex plane with vertex at the origin, $\mathring{A}_{jk}(x,\xi), \mathring{B}_{jk}(x,\xi)$ denote the principal symbols of $A_{jk}(x,D), B_{jk}(x,D)$ respectively. Then the boundary problem (1) will be called elliptic with parameter in $\mathcal{L}$ if **Assumption 1** holds and

- $\det(\mathring{A}(x,\xi) - \eta I) \neq 0$ for $(x,\xi) \in \overline{\Omega} \times \mathbb{R}^n$ and $\eta \in \mathcal{L}$ if $|\xi| + |\eta| \neq 0$.

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- $\det(\mathring{A}(x,\xi) - \eta I) \neq 0$ for $(x,\xi) \in \overline{\Omega} \times \mathbb{R}^n$ and $\eta \in \mathcal{L}$ if $|\xi| + |\eta| \neq 0$.
- for $\xi' \in \mathbb{R}^{n-1}, \eta \in \mathcal{L}$ the problem on the half-line

$$\begin{cases} \mathring{A}(0,\xi',D_n)v(t) - \eta v(t) = 0 \quad \text{for} \quad t = x_n > 0, \\ \mathring{B}_j(0,\xi',D_n)v(t) = 0 \quad \text{at} \quad t = 0, \\ |v(t)| \longrightarrow 0 \quad \text{as} \quad t \longrightarrow \infty \end{cases} \tag{2}$$

has only the trivial solution if $|\xi'| + |\eta| \neq 0$.

Ellipticity with parameter in the sense of Faierman(2006)

- The boundary problem on the line

$$\begin{cases} \mathring{A}(0,\xi',D_n)v^+(t) - \eta v^+(t) = 0 \text{ for } t = x_n > 0, \\ \mathring{A}(0,\xi',D_n)v^-(t) - \eta v^-(t) = 0 \text{ for } t < 0, \\ D_n^l v_j^+(t) - D_n^l v_j^-(t) = 0 \text{ at } t = 0 \text{ for } j = 1,\cdots,N \text{ and} \\ \qquad l = 0,\cdots,m-1, \\ |v^+(t)| \longrightarrow 0 \text{ as } t \longrightarrow \infty, \quad |v^-(t)| \longrightarrow 0 \text{ as } t \longrightarrow -\infty \end{cases}$$

has only the trivial solution for $\xi' \in \mathbb{R}^{n-1}$ and $\eta \in \mathcal{L}$ if $|\xi'| + |\eta| \neq 0$.

Existence, uniqueness, the *a priori* estimates of solutions

Theorem

(Faierman 2006) Suppose (1) *is elliptic with parameter in the sector* $\mathcal{L}$*. Then there is a* $\eta_0 = \eta_0(p) > 0$ *such that for* $\eta \in \mathcal{L}$ *with* $|\eta| \geq \eta_0$, (1) *has a unique solution* $u = (u_1, \cdots, u_N) \in \prod_{j=1}^{N} W_p^{m_j+m}(\Omega)$ *for any* $f = (f_1, \cdots, f_N) \in \prod_{j=1}^{N} W_p^{m_j}(\Omega)$ *and* $g = (g_1, \cdots, g_{mN/2}) \in \prod_{j=1}^{mN/2} W_p^{m-r_j-1/p}(\partial\Omega)$*. The a priori estimate holds for* $C > 0$ *independent of* $f, g,$ *and* η*:*

$$\sum_{j=1}^{N} |||u_j|||_{m_j+m,p} \leq C \left(\sum_{j=1}^{N} |||f_j|||_{m_j,p} + \sum_{j=1}^{mN/2} |||g_j|||_{m-r_j-1/p,p} \right).$$

Part II

Viscous quantum hydrodynamics

Problem

There are several different models simulating the flow of charged particles. For example, quantum energy transport models, the quantum drift diffusion model,the quantum hydrodynamic model(QHD). We study the viscous QHD model which is derived from the Wigner equation with the Fokker-Planck collision operator:

$$\begin{cases} \partial_t n - \mathrm{div} J = \nu_0 \triangle n, \\ \partial_t J - \mathrm{div}\left(\frac{J \otimes J}{n}\right) - T_0 \nabla n + n \nabla V \\ \quad + \frac{\epsilon^2}{2} n \nabla \left(\frac{\triangle \sqrt{n}}{\sqrt{n}}\right) = \nu_0 \triangle J - \frac{J}{\tau}, \\ \lambda^2 \triangle V = n - \mathcal{C}(x). \end{cases} \tag{3}$$

Problem

- initial data

$$(n, J)(0, x) = (n_0, J_0)(x), \quad n_0 \in H^3(\Omega), J_0 \in (H^2(\Omega))^d.$$

- boundary conditions

$$(n, J, V)(t, x)|_{\partial\Omega} = (n_\Gamma, J_\Gamma, V_\Gamma),$$
$$n_\Gamma \in H^{5/2}(\partial\Omega), J_\Gamma \in H^{3/2}(\partial\Omega), V_\Gamma \in H^{3/2}(\partial\Omega).$$

- Compatibility conditions

$$(n_0, J_0)|_{\partial\Omega} = (n_\Gamma, J_\Gamma). \tag{4}$$

$$(\nu_0 \triangle n_0 + \mathrm{div} J_0)|_{\partial\Omega} = 0. \tag{5}$$

Problem

- The unknown functions
  - the particle density n,
  - the current density J,
  - the electrostatic potential V.
- Physical constants (scaled)
  - the temperature T_0,
  - the Planck constant ϵ,
  - the Debye length λ,
  - viscosity constant ν_0,
  - the momentum relaxation time τ,
- a given profile of background charges $\mathcal{C}(x) \in L^2(\Omega)$.

Main Results

Theorem

There is a number $T_* > 0$ *such that the problem* (3) *with given initial data, boundary conditions and compatibility conditions above has a unique local solution on* $[0, T_*)$ *with*

$$n \in L^\infty(0, T_*; H^3(\Omega)), \qquad J \in L^\infty(0, T_*; H^2(\Omega)),$$

$$\partial_t n \in L^2(0, T_*; H^2(\Omega)), \qquad \partial_t J \in L^2(0, T_*; H^1(\Omega)),$$

$$(n, \nabla n, J) \in C([0, T_*) \times \bar{\Omega}), \qquad \partial_t V \in L^2(0, T_*; H^1(\Omega)),$$

$$V \in C(0, T_*; H^2(\Omega)).$$

This solution persists as long as n *stays positive and* $(n, \nabla n, J)$ *are bounded in* $L^\infty(\Omega)$.

Main Results

Theorem

The problem (3) *on a torus* $\mathbb{T}^d$ *with a constant doping profile of background charges* $\mathcal{C}(x) = \mathcal{C}_0 > 0$ *admits a global solution* $(n, J, V) \in H^6(\mathbb{T}^d) \times H^5(\mathbb{T}^d) \times H^4(\mathbb{T}^d)$ *which converges exponentially to the steady state* $(\mathcal{C}_0, 0, 0)$ *provided* $(n_0 - \mathcal{C}_0, J_0) \in H^6(\mathbb{T}^d) \times H^5(\mathbb{T}^d)$ *are sufficiently small and*

$$\inf_{x\in\Omega} n_0(x) > 0, \tag{6}$$

$$\int_\Omega (n_0 - \mathcal{C}_0)dx = 0. \tag{7}$$

Proof of local existence

Steps

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3. obtain approximated solutions P_k of equations for the perturbation and derive uniform bounds of P_k in k in certain senses by induction and via shrinking the time interval;
4. prove the time interval can not tend to null for $k \to \infty$;
5. study the limits of subsequences;
6. prove P_k is a Cauchy sequence in certain sense for sufficiently small time interval then deduce the identity of two limits of any two different convergent subsequences.

Proof of local existence
Step 1:Linearizing

Linearize the system (3) for $U := (n, J)^T$, $J := (J_1, \ldots, J_d)^T$:

$$\partial_t U + A(\partial_x)U + \begin{pmatrix} 0 \\ G \end{pmatrix} = 0, \tag{8}$$

$$A(\partial_x) := \begin{pmatrix} -\nu_0 \triangle & -\mathrm{div} \\ -T_0 \nabla + \frac{\epsilon^2}{4} \nabla \triangle & -\nu_0 \triangle + \tau^{-1} \end{pmatrix},$$

$$G := -\mathrm{div}\left(\frac{J \otimes J}{n}\right) + n \nabla V - \epsilon^2 \mathrm{div}\left((\nabla \sqrt{n}) \otimes (\nabla \sqrt{n})\right).$$

where V_0 solves the Dirichlet problem

$$\begin{cases} \lambda^2 \triangle V_0 = n_0(x) - \mathcal{C}(x), \\ V_0(x)|_{\partial\Omega} = V_\Gamma. \end{cases} \tag{9}$$

Proof of local existence

Step 1:Investigation of $A(\partial_x)$

We shall be concerned with the boundary problem

$$\begin{aligned} A(\partial_x)u(x) - \eta u(x) &= (f^0, f^1, \cdots, f^d)^T \quad \text{in } \Omega, \\ B(x, D)u(x) &= (g^0, g^1, \cdots, g^d)^T \quad \text{on } \partial\Omega, \end{aligned} \tag{10}$$

for $u = (u^0, u^1, \cdots, u^d)$. Consider the equivalent one

$$\begin{aligned} A(x, D)\mathbf{u}(x) - \eta \mathbf{u}(x) &= (f^d, \cdots, f^1, f^0)^T \quad \text{in } \Omega \\ B(x, D)\mathbf{u}(x) &= (g^d, \cdots, g^1, g^0)^T \quad \text{on } \partial\Omega, \end{aligned} \tag{11}$$

for $\mathbf{u} = (u^d, u^{d-1}, \cdots, u^1, u^0)$. Define

$$\begin{aligned} s_1 = s_2 = \cdots = s_d = 2, s_{d+1} = 1, \\ m_1 = m_2 = \cdots = m_d = 0, m_{d+1} = 1, \\ r_1 = r_2 = \cdots = r_d = 0, r_{d+1} = -1. \end{aligned}$$

Proof of local existence

Step 1:Investigation of $A(\partial_x)$

$A(x, D)$ is an $(d+1)\times(d+1)$ matrix operator whose entries $A_{jk}(x, D)$ are linear differential operators defined on Ω of order not exceeding $s_j + m_k$. The $(d+1)\times(d+1)$ matrix operator $B(x, D)$ describes the Dirichlet boundary conditions and is defined as

$$B(x, D) := \begin{pmatrix} I_d & 0 & \cdots & 0 \\ 0 & I_d & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & I_d \end{pmatrix},$$

whose entries are linear differential operators defined on $\partial\Omega$ of order not exceeding $r_j + m_k$ with $r_j < m := s_j + m_j = 2$ and defined to be zero if $r_j + m_k < 0$. Select a sector $\mathcal{L}$ satisfying

Proof of local existence

Step 1:Investigation of $A(\partial_x)$

Lemma

The boundary problem (11) is elliptic with parameter in $\mathcal{L}$ in the sense of M.Faierman(2006) where $\mathcal{L}$ is any closed sector in the complex plane with vertex at the origin such that

$$\mathcal{L} \subset \left\{ \eta \in \mathbb{C}; \quad -\pi + arctg\frac{\epsilon}{2\nu_0} < Arg\eta < -arctg\frac{\epsilon}{2\nu_0} \right\}. \qquad (12)$$

Proof.

$a := \frac{\epsilon^2}{4}|\xi'|^2$, $b := \eta - \nu_0|\xi'|^2$, $\frac{1}{c} := \nu_0^2 + \frac{\epsilon^2}{4}$

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$a := \dfrac{\epsilon^2}{4}|\xi'|^2$, $b := \eta - \nu_0|\xi'|^2$, $\dfrac{1}{c} := \nu_0^2 + \dfrac{\epsilon^2}{4}$

- (i) Check $\det(\mathring{A}(x,\xi) - \eta I) \neq 0$ for $|\eta| + |\xi| \neq 0$ with $\eta \in \mathcal{L}, \xi \in \mathbb{R}^d$.

Proof of local existence

Step 1:Investigation of $A(\partial_x)$

- (ii) Prove that for $\xi' \in \mathbb{R}^{d-1}$ and $\eta \in \mathcal{L}$ the boundary problem on the half-line

$$\begin{cases} \mathring{A}(0,\xi',D_d)v(t) - \eta v(t) = 0 \quad \text{for} \quad t = x_d > 0, \\ v(t) = 0 \quad \text{at} \quad t = 0, \\ |v(t)| \longrightarrow 0 \quad \text{as} \quad t \longrightarrow \infty \end{cases} \tag{13}$$

has only the trivial solution for $\xi' \in \mathbb{R}^{d-1}$ and $\eta \in \mathcal{L}$ if $|\xi'| + |\eta| \neq 0$.

Proof of local existence

Step 1:Investigation of $A(\partial_x)$

Set $v(t) := (v_d(t), v_{d-1}(t), \dots, v_0(t))$ then from (13) we obtain a system of ordinary differential equations with respect to $v(t)$

$$\begin{cases} \frac{\epsilon^2}{4} v_0''' - \nu_0 v_d'' = (\eta - \nu_0|\xi'|^2) v_d + \frac{\epsilon^2}{4}|\xi'|^2 v_0', \\ i\frac{\epsilon^2}{4}\xi_{d-1} v_0'' - \nu_0 v_{d-1}'' = (\eta - \nu_0|\xi'|^2) v_{d-1} + i\frac{\epsilon^2}{4}\xi_{d-1}|\xi'|^2 v_0, \\ \cdot \\ \cdot \\ \cdot \\ i\frac{\epsilon^2}{4}\xi_1 v_0'' - \nu_0 v_1'' = (\eta - \nu_0|\xi'|^2) v_1 + i\frac{\epsilon^2}{4}\xi_1|\xi'|^2 v_0, \\ -v_d' - i\xi_{d-1} v_{d-1} \cdots - i\xi_1 v_1 = (\eta - \nu_0|\xi'|^2) v_0 + \nu_0 v_0''. \end{cases} \tag{14}$$

Proof of local existence

Step 1:Investigation of $A(\partial_x)$

Define $y := \sum_{j=1}^{d-1} \xi_j v_j$, then $(v_0, v_d, y, v_0', v_0'', y')^T$ solves the following system of first-order:

$$\mathbf{y}' = A\mathbf{y}, \tag{15}$$

$$A := \begin{pmatrix} 0 & 0 & 0 & 1 & 0 & 0 \\ -b & 0 & -i & 0 & -\nu_0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & bc & 0 & ac - \nu_0 bc & 0 & -i\nu_0 c \\ -i\frac{a}{\nu_0}|\xi'|^2 & 0 & -\frac{b}{\nu_0} & 0 & i\frac{a}{\nu_0} & 0 \end{pmatrix}. \tag{16}$$

Proof of local existence

Step 1:Investigation of $A(\partial_x)$

$v_0(t)$ solves

$$\left(\nu_0^2+\frac{\epsilon^2}{4}\right)v_0^{(4)}+\left(2\nu_0(\eta-\nu_0|\xi'|^2)-\frac{\epsilon^2}{2}|\xi'|^2\right)v_0'' + \left(\frac{\epsilon^2}{4}|\xi'|^4+(\eta-\nu_0|\xi'|^2)^2\right)v_0=0. \tag{17}$$

Case 1: $\eta=0$.
$|\xi'|+|\eta|\neq 0 \Longrightarrow |\xi'|\neq 0$

$$v_0(t)=C_{01}e^{|\xi'|t}+C_{02}te^{|\xi'|t}+C_{03}e^{-|\xi'|t}+C_{04}te^{-|\xi'|t},$$

$|v_0|\longrightarrow 0$ for $t\longrightarrow\infty$ and $v_0(0)=0\Longrightarrow$

$$v_0(t)=C_{04}te^{-|\xi'|t}. \tag{18}$$

Proof of local existence

Step 1:Investigation of $A(\partial_x)$

$$A \sim \begin{pmatrix} |\xi'| & 1 & 0 & 0 & 0 & 0 \\ 0 & |\xi'| & 0 & 0 & 0 & 0 \\ 0 & 0 & |\xi'| & 0 & 0 & 0 \\ 0 & 0 & 0 & -|\xi'| & 1 & 0 \\ 0 & 0 & 0 & 0 & -|\xi'| & 0 \\ 0 & 0 & 0 & 0 & 0 & -|\xi'| \end{pmatrix}.$$

$$v_d = C_d t e^{-|\xi'|t}, \tag{19}$$

$$y = C_y t e^{-|\xi'|t}. \tag{20}$$

Substituting into the system yields $v(t) = 0$

Proof of local existence

Step 1:Investigation of $A(\partial_x)$

Case 2: $\eta \neq 0$.

v_0 is a complex linear combination of $e^{\lambda_1 t}, e^{\lambda_2 t}, e^{-\lambda_1 t}, e^{-\lambda_2 t}$:

$$v_0(t) = c_{01}e^{-\lambda_1 t} + c_{02}e^{-\lambda_2 t} + c_{03}e^{\lambda_1 t} + c_{04}e^{\lambda_2 t}.$$

$\lambda_1, \lambda_2, -\lambda_1, -\lambda_2$ with $\mathrm{Re}\lambda_1 > 0$, $\mathrm{Re}\lambda_2 > 0$ are 4 different eigenvalues of A, which is similar as

$$\begin{pmatrix} \lambda_1 & 0 & 0 & 0 & 0 & 0 \\ 0 & \lambda_2 & 0 & 0 & 0 & 0 \\ 0 & 0 & -\lambda_1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -\lambda_2 & 0 & 0 \\ 0 & 0 & 0 & 0 & \sqrt{-\frac{b}{\nu_0}} & 0 \\ 0 & 0 & 0 & 0 & 0 & -\sqrt{-\frac{b}{\nu_0}} \end{pmatrix}.$$

Proof of local existence

Step 1:Investigation of $A(\partial_x)$

$$\begin{cases} v_0 = c_{01}e^{-\lambda_1 t} + c_{02}e^{-\lambda_2 t}, \\ v_d = c_{d1}e^{-\lambda_1 t} + c_{d2}e^{-\lambda_2 t} + c_{d3}e^{-\lambda_3 t}, \\ y = c_{y1}e^{-\lambda_1 t} + c_{y2}e^{-\lambda_2 t} + c_{y3}e^{-\lambda_3 t}. \end{cases} \tag{21}$$

Since $v(0) = 0$ we find

$$\begin{cases} c_{01} + c_{02} = 0, \\ c_{d1} + c_{d2} + c_{d3} = 0, \\ c_{y1} + c_{y2} + c_{y3} = 0. \end{cases} \tag{22}$$

Substituting into the system yields $v(t) = 0$.

Proof of local existence

Step 1:Investigation of $A(\partial_x)$

- (iii) The boundary problem on the line ($l = 0, 1$)

$$\begin{cases} \mathring{A}(0,\xi',D_d)v^+(t) - \eta v^+(t) = 0 \quad \text{for} \quad t = x_d > 0, \\ \mathring{A}(0,\xi',D_d)v^-(t) - \eta v^-(t) = 0 \quad \text{for} \quad t < 0, \\ D_d^l v_j^+(t) - D_d^l v_j^-(t) = 0 \quad \text{at} \quad t = 0 \quad \text{for} \quad j = d,\cdots,0, \\ |v^+(t)| \longrightarrow 0 \quad \text{as} \quad t \longrightarrow \infty, \quad |v^-(t)| \longrightarrow 0 \quad \text{as} \quad t \longrightarrow -\infty \end{cases} \tag{23}$$

has only the trivial solution for $\xi' \in \mathbb{R}^{d-1}$ and $\eta \in \mathcal{L}$ if $|\xi'| + |\eta| \neq 0$.

$\square$

Proof of local existence

Step 2: existence theorem for the linear Problem

The linear problem

$$\begin{cases} \partial_t \mathbf{u}(t,x) + A(\partial_x)\mathbf{u}(t,x) = F(t,x), \quad (t,x) \in [0,T) \times \Omega \\ \mathbf{u}(0,x) = 0, \\ \mathbf{u}\big|_{\partial\Omega} = 0 \quad \text{for a.e. } t \in [0,T]. \end{cases} \tag{24}$$

with $F(t,x) := (F^0, F^{\mathbf{d}}), F^{\mathbf{d}} := (F^1, \cdots, F^d)$

$$\begin{cases} F^0 \in L^\infty(0,T; H^1(\Omega)), & F^{\mathbf{d}} \in L^\infty(0,T; (L^2(\Omega))^d); \\ \dot{F}^0 \in L^\infty(0,T; L^2(\Omega)), & \dot{F}^{\mathbf{d}} \in L^\infty(0,T; (H^{-1}(\Omega))^d); \\ F^0(0,x) \in H_0^1(\Omega), & F^{\mathbf{d}}(0,x) \in L^2(\Omega), \end{cases} \tag{25}$$

Proof of local existence

Step 2: existence theorem for the linear Problem

has a unique solution $\mathbf{u}(t,x) := (u^0, u^{\mathbf{d}}), u^{\mathbf{d}} := (u^1, \cdots, u^d)$ with

$$\begin{cases} u^0 \in L^\infty(0,T;H^3(\Omega)) \cap C([0,T], C^1(\overline{\Omega})) \cap C([0,T], H^2(\Omega)), \\ u^{\mathbf{d}} \in L^\infty(0,T;H^2(\Omega)) \cap C([0,T], C(\overline{\Omega})) \cap C([0,T], H^1(\Omega)), \\ \partial_t u^0 \in L^\infty(0,T;H^1(\Omega)) \cap C([0,T], L^2(\Omega)) \cap L^2(0,T;H^2(\Omega)), \\ \partial_t u^{\mathbf{d}} \in L^\infty(0,T;L^2(\Omega)) \cap C([0,T], L^2(\Omega)) \cap L^2(0,T;H^1(\Omega)), \end{cases}$$

which satisfies the a priori estimates

$$\|u^0\|^2_{L^\infty(0,T;H^3(\Omega))} + \|u^{\mathbf{d}}\|^2_{L^\infty(0,T;H^2(\Omega))} \leq C = C(F,T).$$

Proof of local existence

Step 3: Uniform bounds of P_k in a fixed time interval

We study the equations for the perturbation $P := (P^0, P^{\mathbf{d}}) := (n - n_0, J - J_0)$ and construct approximate solutions $P_k := (P_k^0, P_k^{\mathbf{d}})$ for $P_k^0 := n_k - n_0,\ P_k^{\mathbf{d}} := J_k - J_0$ $(k \geq 1)$:

$$\begin{cases} \partial_t P_k + A(\partial_x) P_k = F_{k-1}, \\ P_k(0,x) = 0, \\ P_k(t,x) = 0, \quad \text{on } \partial\Omega \text{ for a.e. } 0 \leq t \leq T, \end{cases} \tag{26}$$

Proof of local existence

Step 3: Uniform bounds of P_k in a fixed time interval

$$F_{k-1} := -\begin{pmatrix} 0 \\ S_{k-1} \end{pmatrix} - A(\partial_x)\begin{pmatrix} n_0 \\ J_0 \end{pmatrix} \ (k \geq 2),$$

$$S_{k-1} := \operatorname{div}\left(\frac{(P^{\mathbf{d}}_{k-1} + J_0) \otimes (P^{\mathbf{d}}_{k-1} + J_0)}{P^0_{k-1} + n_0}\right) - (P^0_{k-1} + n_0)\nabla V_{k-1}$$
$$+ \epsilon^2 \operatorname{div}\left(\left(\nabla\sqrt{P^0_{k-1} + n_0}\right) \otimes \left(\nabla\sqrt{P^0_{k-1} + n_0}\right)\right),$$

$$\lambda^2 \triangle V_{k-1} = P^0_{k-1} + n_0 - \mathcal{C}(x), \quad V_{k-1}(t,x)|_{\partial\Omega} = V_\Gamma(x).$$

Proof of local existence

Step 3: Uniform bounds of P_k in a fixed time interval

Let $k \geq 0$ and $P_0 = 0$, there exist $t' > 0$, $C_0 > 0$, $C^* > 0$ and $C' > 0$ independent of k such that in the interval $[0, t']$ P_k satisfies

$$\begin{cases} P_k^0 \in L^\infty(0, t'; H^3(\Omega)) \cap C([0, t'], C^1(\overline{\Omega})) \cap C([0, t'], H^2(\Omega)), \\ P_k^{\mathbf{d}} \in L^\infty(0, t'; H^2(\Omega)) \cap C([0, t'], C(\overline{\Omega})) \cap C([0, t'], H^1(\Omega)), \\ \partial_t P_k^0 \in L^\infty(0, t'; H^1(\Omega)) \cap C([0, t'], L^2(\Omega)) \cap L^2(0, t'; H^2(\Omega)), \\ \partial_t P_k^{\mathbf{d}} \in L^\infty(0, t'; L^2(\Omega)) \cap C([0, t'], L^2(\Omega)) \cap L^2(0, t'; H^1(\Omega)), \end{cases} \tag{27}$$

Proof of local existence

Step 3: Uniform bounds of P_k in a fixed time interval

$$\begin{cases} \|P_k^0\|_{L^\infty(0,t';H^3(\Omega))} \le C', & \|P_k^{\mathbf{d}}\|_{L^\infty(0,t';H^2(\Omega))} \le C', \\ \|P_k^0\|_{L^\infty(0,t';H^2(\Omega))} \le C^*, & \|P_k^{\mathbf{d}}\|_{L^\infty(0,t';H^1(\Omega))} \le C^*, \\ \|P_k^0\|_{L^\infty(0,t';H^1(\Omega))} \le C_0, & \|P_k^{\mathbf{d}}\|_{L^\infty(0,t';L^2(\Omega))} \le C_0, \\ \|\dot{P}_k^0\|_{L^\infty(0,t';H^1(\Omega))} \le C_0, & \|\dot{P}_k^{\mathbf{d}}\|_{L^\infty(0,t';L^2(\Omega))} \le C_0, \\ \|\dot{P}_k^0\|_{L^2(0,t';H^2(\Omega))} \le C_0, & \|\dot{P}_k^{\mathbf{d}}\|_{L^2(0,t';H^1(\Omega))} \le C_0, \end{cases} \quad (28)$$

$$\begin{cases} \inf_{[0,t']} \inf_{x\in\overline{\Omega}} (P_k^0 + n_0) > \delta_0, \\ \sup_{[0,t']} \max \left(\|\nabla(P_k^0 + n_0)\|_{L^\infty(\Omega)}, \|P_k^0 + n_0\|_{L^\infty(\Omega)}, \|P_k^{\mathbf{d}} + J_0\|_{L^\infty(\Omega)} \right) \\ < \delta_0^{-1}. \end{cases} \quad (29)$$

Proof of local existence

Step 3: Uniform bounds of P_k in a fixed time interval

Use mathematical induction to complete the proof.

- $k = 0$, $P_0 = 0$;

Proof of local existence

Step 3: Uniform bounds of P_k in a fixed time interval

Use mathematical induction to complete the proof.

- $k = 0$, $P_0 = 0$;
- there exists a time interval $[0, t^*]$ such that suppose P_{k-1} satisfies the assumptions above on $[0, t^*]$, then P_k satisfies (27) and (28);

Proof of local existence

Step 3: Uniform bounds of P_k in a fixed time interval

Use mathematical induction to complete the proof.

- $k = 0,\ P_0 = 0$;
- there exists a time interval $[0, t^*]$ such that suppose P_{k-1} satisfies the assumptions above on $[0, t^*]$, then P_k satisfies (27) and (28);
- shrink $[0, t^*]$ into $[0, t_k]$ such that P_k satisfies (27), (28) and (29) on $[0, t_k]$;

Proof of local existence
Step 3: Uniform bounds of P_k in a fixed time interval

Use mathematical induction to complete the proof.

- $k = 0$, $P_0 = 0$;
- there exists a time interval $[0, t^*]$ such that suppose P_{k-1} satisfies the assumptions above on $[0, t^*]$, then P_k satisfies (27) and (28);
- shrink $[0, t^*]$ into $[0, t_k]$ such that P_k satisfies (27), (28) and (29) on $[0, t_k]$;
- $[0, t_k]$ can not tend to zero from Sobolev imbedding theorem and interpolation inequalities.

Proof of local existence

Step 3: Uniform bounds of P_k in a fixed time interval

There exists a subsequence of (P_k) for $\delta > 0$

$$P_k^0 \to P^0 \text{ in } C([0,t'],H^{3-\delta}(\Omega));\, P_k^{\mathbf{d}} \to P^{\mathbf{d}} \text{ in } C([0,t'],H^{2-\delta}(\Omega)).$$

$$(P_k^0, \nabla P_k^0, P_k^{\mathbf{d}}) \to (P^0, \nabla P^0, P^{\mathbf{d}}) \quad \text{in} \quad C([0,t'] \times \bar{\Omega}),$$

$$\dot{P}_k^0 \rightharpoonup \dot{P}^0 \quad \text{in } L^2(0,t';H^2(\Omega)), \dot{P}_k^{\mathbf{d}} \rightharpoonup \dot{P}^{\mathbf{d}} \text{ in } L^2(0,t';H^1(\Omega)),$$
$$P_k^0 \rightharpoonup^* P^0 \quad \text{in } L^\infty(0,t';H^3(\Omega)), P_k^{\mathbf{d}} \rightharpoonup^* P^{\mathbf{d}} \text{ in } L^\infty(0,t';H^2(\Omega)).$$

Proof of local existence

Step 3: Cauchy sequence

$$\|P^0_{k+1} - P^0_k\|^2_{L^\infty(0,t';H^1(\Omega))} + \|P^{\mathbf{d}}_{k+1} - P^{\mathbf{d}}_k\|^2_{L^\infty(0,t';L^2(\Omega))}$$
$$\leq Ct'\left(\|P^0_k - P^0_{k-1}\|^2_{L^\infty(0,t';H^1(\Omega))} + \|P^{\mathbf{d}}_k - P^{\mathbf{d}}_{k-1}\|^2_{L^\infty(0,t';L^2(\Omega))}\right).$$

For small $0 < T_* \leq t'$, the total sequence (P_k) converges to P^* on $[0, T_*]$ in $L^\infty(0, T_*; H^1(\Omega)) \times L^\infty(0, T_*; L^2(\Omega))$, which implies

$$P = P^*.$$

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